

Fractional Differential Equations with Periodic Boundary Conditions of Constant Ratio

Anwarrud Din

Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, China

E-mail: anwarm.phil@yahoo.com

Shah Faisal

Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan

E-mail: shahfaisal8763@gmail.com

Abstract

This article is concerned with the existence and uniqueness of solutions to some fractional order boundary value problems of the type

$$\begin{aligned} {}^c D_{0+}^\alpha u(t) &= f(t, u(t), {}^c D^{\alpha-1} u(t)), \quad 1 < \alpha \leq 2, t \in J = [0, 1] \\ u(0) &= \xi u(1), \quad {}^c D^\beta u(0) = \xi {}^c D^\beta u(1), \quad 0 < \beta < 1, \quad \xi \in (0, 1), \end{aligned}$$

where ${}^c D_{0+}^\alpha$ represents Caputo fractional order derivative and $f : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous function explicitly depending on fractional order derivative. Our results are based on some fixed point theorems of functional analysis. For the applicability of our results, we provide an example.

Keywords. Fractional order differential equations; Boundary value problems; Positives solution; Uniqueness result.

Mathematics Subject Classification: 34A08, 34B37, 35R11

1 Introduction

In the last few decades it was realized that in some real world problems, fractional order models are more adequate and accurate compared to their counterpart integer-order models. The main advantage of fractional models in comparison with classical integer models is that it provide an excellent tool for description of memory and heredity properties of various materials and processes. This theory has many application in different field of sciences like engineering, physics, chemistry, biology etc, see for example [4–10] and reference therein. In consequence, the subject of fractional order differential equations gaining much importance and attentions. But it should be noted that most of the available literature are devoted to the solvability of linear initial value problems for fractional order differential equation in term of special functions [3–6].

Existence theory corresponding to boundary value problems for fractional order differential equations had attracted the attention of researchers quite recently. There are some work dealing with the existence and multiplicity of positive solution to non-linear initial value problems associated with fractional order differential equation, [7–10]. Recently, M. Benchohra and N. Hamidi and J. Henderson [11] studied non-linear fractional order differential equation with periodic boundary. However, the problem with the boundary conditions we study had never been studied before. We are concern with the existence and uniqueness of positive solutions to boundary value problem of the form

$$\begin{aligned} {}^c D_{0+}^\alpha u(t) &= f(t, u(t), {}^c D^{\alpha-1} u(t)), \quad 1 < \alpha \leq 2, t \in J = [0, 1] \\ u(0) &= \xi u(1), \quad {}^c D^\beta u(0) = \xi {}^c D^\beta u(1), \quad 0 < \beta < 1, \quad \xi \in (0, 1). \end{aligned} \tag{1.1}$$

We use results from the functional analysis. We recall some notations, definition and necessary lemmas [1–3] necessary for our investigation. The Banach space of all continuous functions from $J = [0, 1]$ into \mathbb{R} with the norm $\|y\|_\infty = \sup\{|y(t)|; 0 \leq t \leq 1\}$ is denoted by $C(J, \mathbb{R})$ and Banach space of Lebesgue integrable function with the norm $\|y\|_{L^1} = \int_0^1 |y(t)|dt$ is denoted by $L^1(J, \mathbb{R})$. Let us denote by $\tilde{C}(J, \mathbb{R}) = \{u \in C(J, \mathbb{R}), {}^c D^{\alpha-1}u \in C(J, \mathbb{R})\}$, then $\tilde{C}(J, \mathbb{R})$ is a Banach space under the norm $\|u\|_{\tilde{C}} = \max\{\|u\|_\infty, \|{}^c D^{\alpha-1}u\|_\infty\}$.

Definition 1.1. The fractional order integral of the function $h \in l^1(J, \mathbb{R})$ of order $\alpha \in \mathbb{R}$, is defined by

$$I_0^\alpha f(t) = \frac{1}{\Gamma\alpha} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where Γ is the gamma function.

Definition 1.2. For a function f the α th order Caputo fractional derivative is define by

$$({}^c D_{0+}^\alpha)f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,$$

where n is integer such that $n = \lceil \alpha \rceil$.

Lemma 1.3. Let $\alpha > 0$ and $u \in C(0, 1) \cap L(0, 1)$, the fractional order differential equation

$$D_{0+}^\alpha u(t) = 0, \quad n-1 < \alpha < n$$

has a unique solution of the form

$$u(t) = C_0 + C_1 t + \dots + C_{n-1} t^{n-1}, \quad C_i \in \mathbb{R}, \quad i = 1, 2, 3, \dots, n-1.$$

Lemma 1.4. Let $\alpha > 0$ then

$$I^\alpha {}^c D_{0+}^\alpha u(t) = u(t) + C_0 + C_1 t + \dots + C_{n-1} t^{n-1}, \quad \text{for } C_i \in \mathbb{R}, \quad i = 1, 2, 3, \dots, n-1, \alpha \leq n < \alpha+1$$

2 Main results

Now we study sufficient conditions for existence and uniqueness of solutions.

Lemma 2.1. For $y \in C(J)$, the linear fractional order boundary value problem

$$\begin{aligned} {}^c D^\alpha u(t) &= y(t), \quad 1 < \alpha \leq 2, \quad t \in J = [0, 1] \\ u(0) &= \xi u(1), \quad {}^c D^\beta u(0) = \xi {}^c D^\beta u(1), \quad 0 < \beta < 1 \quad \xi \in (0, 1) \end{aligned}$$

has a solution of the form $u(t) = \int_0^1 G(t, s)y(s)ds$, where the Green function is

$$G(t, s) = \begin{cases} \frac{(1-\xi)(t-s)^{\alpha-1} + \xi(1-s)^{\alpha-1}}{\Gamma\alpha(1-\xi)} - \frac{\Gamma(2-\beta)(\xi+(1-\xi)t)}{\Gamma(\alpha-\beta)(1-\xi)}(1-s)^{\alpha-\beta-1}, & \text{if } 0 \leq s \leq t \leq 1 \\ \frac{\xi(1-s)^{\alpha-1}}{\Gamma\alpha(1-\xi)} - \frac{\Gamma(2-\beta)(\xi+(1-\xi)t)}{\Gamma(\alpha-\beta)(1-\xi)}(1-s)^{\alpha-\beta-1}, & \text{if } 0 \leq t \leq s \leq 1, \end{cases}$$

Proof. By lemma (1.4), solution of the fractional order differential equation ${}^c D^\alpha u(t) = y(t)$ is given by

$$u(t) = I^\alpha y(t) + C_0 + C_1 t \text{ and } {}^c D^\beta u(t) = I^{\alpha-\beta} y(t) + C_1 \frac{t^{1-\beta}}{\Gamma(2-\beta)}.$$

Using the boundary conditions $u(0) = \xi u(1)$ and ${}^c D^\beta u(0) = \xi {}^c D^\beta u(1)$, we obtain

$$C_0 = \frac{\xi}{1-\xi} I^\alpha y(1) - \Gamma(2-\beta) \frac{\xi}{1-\xi} I^{\alpha-\beta} y(1), \text{ and } C_1 = -\Gamma(2-\beta) I^{\alpha-\beta}.$$

Hence $u(t) = I^\alpha y(t) + \frac{\xi}{1-\xi} I^\alpha y(1) - \Gamma(2-\beta)(\frac{\xi}{1-\xi} + t) I^{\alpha-\beta} y(1)$ which implies that

$$\begin{aligned} u(t) &= \frac{1}{\Gamma\alpha} \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{\xi}{1-\xi} \int_0^1 (1-s)^{\alpha-1} y(s) ds \\ &\quad - \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)} \left(t + \frac{\xi}{1-\xi}\right) \int_0^1 (1-s)^{\alpha-\beta-1} y(s) ds = \int_0^1 G(t,s) y(s) ds. \end{aligned}$$

□

If $\alpha-\beta < 1$, the Green function $G(t,s)$ become unbounded but the function $t \rightarrow \int_0^1 G(t,s) y(s) ds$ is continuous on J so attain its spermium value say $G^* = \sup_{t \in J} \int_0^1 |G(t,s)| ds$. In view of Lemma (2.1), an equivalent representation of the BVP (1.1) is given by

$$u(t) = \int_0^1 G(t,s) f(s, u(s), {}^c D^{\alpha-1} u(s)) ds, \quad t \in J. \quad (2.1)$$

$T : \tilde{C}(J, \mathbb{R}) \rightarrow \tilde{C}(J, \mathbb{R})$ by

$$Tu(t) = \int_0^1 G(t,s) f(s, u(s), {}^c D^{\alpha-1} u(s)) ds, \quad (2.2)$$

then solutions of the BVP (1.1) are fixed points of T .

Theorem 2.2. Assume that $f : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and the following hold
(A₁) there exist $p \in C(J, \mathbb{R}^+)$ and a continuous, non-decreasing function $\psi : [0, \infty] \rightarrow (0, \infty)$ such that

$$|f(t, u, z)| \leq p(t) \psi(|z|) \quad \text{for all } t \in J, u, z \in \mathbb{R}.$$

(A₂) there exist constant $r > 0$ such that

$$r \geq \max \left\{ G^* p^* \psi(r), p^* \psi(r) \frac{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1) + \Gamma(2-\beta)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)} \right\},$$

where $p^* = \sup \{p(s), s \in J\}$,

then the BVP (1.1) has at lest one solution such that $|u(t)| < r$ on J .

Proof. We prove the result via Schauder fixed point theorem. Firstly, we prove that the operator T defined by (2.2) is continuous. Choose r as in (A₂) and define $D = \{u \in \tilde{C}(J, \mathbb{R}), \|u\|_{\tilde{C}} \leq r\}$ a closed and bounded subset of $\tilde{C}(J, \mathbb{R})$. Let the sequence $\{u_n\}$ converges to u in $\tilde{C}(J, \mathbb{R})$, then as in the proof of Lemma 2, 5 of [11], $D^{\alpha-1} u_n \rightarrow D^{\alpha-1} u$. Choose $\rho > 0$ such that

$$\|u_n\|_{\tilde{C}} \leq \rho, \quad \|u\|_{\tilde{C}} \leq \rho.$$

For for all $t \in J$, we have

$$|Tu_n(t) - Tu(t)| \leq \int_0^1 |G(t,s) [f(s, u_n(s), D^{\alpha-1} u_n(s)) - f(s, u(s), D^{\alpha-1} u(s))]| ds \text{ and}$$

$$\begin{aligned} |D^{\alpha-1} Tu_n(t) - D^{\alpha-1} Tu(t)| &\leq \int_0^t |f(s, u_n(s), D^{\alpha-1} u_n(s)) - f(s, u(s), D^{\alpha-1} u(s))| ds \\ &+ \frac{\Gamma(2-\beta)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta)} t^{2-\alpha} \int_0^1 (1-s)^{\alpha-\beta-1} |f(s, u_n(s), D^{\alpha-1} u_n(s)) - f(s, u(s), D^{\alpha-1} u(s))| ds. \end{aligned}$$

From the continuity of $f(s, u(s), D^{\alpha-1} u(s))$ and Lebesgue dominated convergence theorem, it follows that

$$\|Tu_n(t) - Tu(t)\|_{\infty} \rightarrow 0 \text{ and } \|D^{\alpha-1} Tu_n(t) - D^{\alpha-1} Tu(t)\|_{\infty} \rightarrow 0$$

as $n \rightarrow \infty$ which implies that T is continuous.

Now we show that $TD \subseteq D$. Let $u \in D$ then for each $t \in J$, we have

$$\begin{aligned} |Tu(t)| &\leq \int_0^1 (|G(t, s)| |f(s, u(s), {}^c D^{\alpha-1} u(s))|) ds \leq G^* \int_0^1 (|f(s, u(s), {}^c D^{\alpha-1} u(s))|) ds \\ &\leq G^* p^* \psi(\|u\|_{\tilde{c}}) \leq G^* p^* \psi(r), \end{aligned}$$

$$\begin{aligned} |{}^c D^{\alpha-1} Tu(t)| &\leq \int_0^t |f(s, u(s), {}^c D^{\alpha-1} u(s))| ds \\ &\quad + \frac{\Gamma(2-\beta)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta)} t^{2-\alpha} \int_0^1 (1-s)^{\alpha-\beta-1} |f(s, u(s), {}^c D^{\alpha-1} u(s))| ds \\ \|{}^c D^{\alpha-1} Tu(t)\|_{\infty} &\leq P^* \psi(\|u\|_{\tilde{c}}) \left(\frac{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1) + \Gamma(2-\beta)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)} \right). \end{aligned}$$

Consequently,

$$\|Tu(t)\|_{\tilde{c}} \leq \max \left\{ G^* p^* \psi(r), p^* \psi(r) \left(\frac{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1) + \Gamma(2-\beta)}{\Gamma(2-\beta+1)\Gamma(3-\alpha)} \right) \right\} \leq r$$

implies that $Tu(t) \in D$ for all $u(t) \in D$.

Finally, we show that T maps D into equicontinuous set of $\tilde{C}(J, \mathbb{R})$. Take $t_1, t_2 \in J$ such that $t_1 < t_2$ and $u \in D$, we have

$$\begin{aligned} |Tu(t_2) - Tu(t_1)| &\leq \int_0^1 |G(t_2, s) - G(t_1, s)| |f(s, u(s), {}^c D^{\alpha-1} u(s))| ds \\ &\leq p^* \psi(\|u\|_{\tilde{c}}) \int_0^1 |G(t_2, s) - G(t_1, s)| ds, \end{aligned}$$

In view of the continuity of G , it follows that

$$\|Tu(t_2) - Tu(t_1)\| \rightarrow 0 \text{ as } t_2 \rightarrow t_1.$$

Further,

$$\begin{aligned} |(T^c D^{\alpha-1} u)(t_2) - (T^c D^{\alpha-1} u)(t_1)| &\leq \left| \int_0^{t_2} f(s, u(s), {}^c D^{\alpha-1} u(s)) ds - \int_0^{t_1} f(s, u(s), {}^c D^{\alpha-1} u(s)) ds \right| \\ &\quad + \frac{(t_1^{2-\alpha} - t_2^{2-\alpha})\Gamma(2-\beta)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta)} \left| \int_0^1 (1-s)^{\alpha-\beta-1} f(s, u(s), {}^c D^{\alpha-1} u(s)) ds \right| \\ &\leq p^* \psi(\max(\|u\|_{\infty}, \|{}^c D^{\alpha-1} u(t)\|_{\infty})) \left\{ t_2 - t_1 + \frac{(t_1^{2-\alpha} - t_2^{2-\alpha})\Gamma(2-\beta)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)} \right\} \\ &\leq p^* \psi(\|u\|_{\tilde{c}}) \left\{ t_2 - t_1 + \frac{(t_1^{2-\alpha} - t_2^{2-\alpha})\Gamma(2-\beta)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)} \right\}, \end{aligned}$$

which implies that

$$|T^c D^{\alpha-1} u(t_2) - (T^c D^{\alpha-1} u)(t_1)| \rightarrow 0 \text{ as } t_2 \rightarrow t_1.$$

Hence

$$\|(Tu)_{t_2} - (Tu)_{t_1}\|_{\tilde{C}} = \max \left\{ \|(Tu)_{t_2} - (Tu)_{t_1}\|_{\infty}, \|T(cD^{\alpha-1} u)_{t_2} - (TcD^{\alpha-1} u)_{t_1}\|_{\infty} \right\} \rightarrow 0 \text{ as } t_2 \rightarrow t_1.$$

By Arzela-Ascoli Theorem, T is completely continuous and by Schauder fixed point theorem, T has a fixed point u in D such that $|u(t)| < r$ for all $t \in J$. \square

Theorem 2.3. Assume that $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exist a constant $k > 0$ such that

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq k(|u - \bar{u}| + |v - \bar{v}|), \text{ for } t \in J \text{ and } u, v, \bar{u}, \bar{v} \in \mathbb{R}.$$

If

$$\max\{2G^*k, 2k \frac{(\Gamma(3-\alpha)(\Gamma(\alpha-\beta+1) + \Gamma(2-\beta))}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)}\} < 1$$

then the BVP (1.1) has a unique solution.

Proof. The proof is based on Banach Fixed point theorem, we show that the operator T is contraction. For $u, \bar{u} \in \tilde{C}(J, \mathbb{R})$ and $t \in J$, we have

$$\begin{aligned} |(Tu)t - (T\bar{u})t| &\leq \sup_{t \in J} \int_0^1 G(t, s) |f(s, u(s), {}^c D^{\alpha-1}u(s)) - f(s, \bar{u}(s), {}^c D^{\alpha-1}\bar{u}(s))| ds \\ &\leq G^*k\{|u - \bar{u}| + |D^{\alpha-1}u - D^{\alpha-1}\bar{u}|\} \leq 2G^*k \|u - \bar{u}\|_{\tilde{C}} \end{aligned}$$

and

$$\begin{aligned} |D^{\alpha-1}Tu(t) - D^{\alpha-1}T\bar{u}(t)| &\leq \int_0^t |f(s, u(s), D^{\alpha-1}u(s)) - f(s, \bar{u}(s), D^{\alpha-1}\bar{u}(s))| ds \\ &+ \frac{\Gamma(2-\beta)t^{2-\alpha}}{\Gamma(3-\beta)\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} |f(s, u(s), D^{\alpha-1}u(s)) - f(s, \bar{u}(s), D^{\alpha-1}\bar{u}(s))| ds \\ &\leq (2k + \frac{2k\Gamma(2-\beta)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)})\|u - \bar{u}\|_{\tilde{C}} \leq 2k(\frac{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1) + \Gamma(2-\beta)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)})\|u - \bar{u}\|_{\tilde{C}}. \end{aligned}$$

Since

$$\max\left\{2G^*k, 2k\left(\frac{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1) + \Gamma(2-\beta)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)}\right)\right\} = d < 1$$

it follows that

$$\|Tu(t) - T\bar{u}(t)\|_c \leq d \|u - \bar{u}\|_{\tilde{C}}$$

and by Banach fixed point theorem, the B.V.P (1.1) has a unique solution. \square

Example 2.4. Consider the fractional boundary value problem

$${}^c D^{\frac{3}{2}}u(t) = \frac{\sin^2 t}{11(e^{2t} + 3e^t + 1)} \left(3 + t + 5u(t) + D^{\frac{1}{2}}u(t)\right),$$

where $u(0) = \frac{1}{2}u(1)$, ${}^c D^{\frac{1}{2}}u(0) = \frac{1}{2}$, ${}^c D^{\frac{1}{2}}u(1) = \frac{1}{2}$ for $\xi = \beta = \frac{1}{2}$.

Here

$$f(t, u(t), D^{\frac{1}{2}}u(t)) = \frac{\sin^2 t}{11(e^{2t} + 3e^t + 1)} \left(3 + t + 5u(t) + D^{\frac{1}{2}}u(t)\right),$$

therefore

$$\begin{aligned} |f(t, u(t), D^{\frac{1}{2}}u(t)) - f(t, \bar{u}(t), D^{\frac{1}{2}}\bar{u}(t))| &\leq \frac{1}{11}(|u - \bar{u}| + |D^{\frac{1}{2}}u - D^{\frac{1}{2}}\bar{u}|) \\ &\leq k(|u - \bar{u}| + |v - \bar{v}|) \end{aligned}$$

which is the condition (A_1) with $k = \frac{1}{11}$.

Also

$$G(t, s) = \begin{cases} \frac{(1-\xi)(t-s)^{\alpha-1} + \xi(1-s)^{\alpha-1}}{\Gamma(\alpha(1-\xi))} - \frac{\Gamma(2-\beta)(\xi + (1-\xi)t)}{\Gamma(\alpha-\beta)(1-\xi)}(1-s)^{\alpha-\beta-1}, & \text{if } 0 \leq s \leq t \leq 1 \\ \frac{\xi(1-s)^{\alpha-1}}{\Gamma(\alpha(1-\xi))} - \frac{\Gamma(2-\beta)(\xi + (1-\xi)t)}{\Gamma(\alpha-\beta)(1-\xi)}(1-s)^{\alpha-\beta-1}, & \text{if } 0 \leq t \leq s \leq 1, \end{cases}$$

Or

$$G(t, s) = \left\{ \begin{array}{l} \frac{\frac{1}{2}(t-s)^{\frac{1}{2}} + \frac{1}{2}(1-s)^{\frac{1}{2}}}{\frac{1}{2}\Gamma(\frac{3}{2})} - \frac{\Gamma(\frac{3}{2})(\frac{1}{2} + \frac{1}{2}t)}{\frac{1}{2}\Gamma(1)} \\ \frac{\frac{1}{2}(1-s)^{\frac{1}{2}}}{\frac{1}{2}\Gamma(\frac{3}{2})} - \frac{(\frac{1}{2} + \frac{1}{2}t)\Gamma(\frac{3}{2})}{\frac{1}{2}\Gamma(1)} \end{array} \right.$$

so that

$$G^* \leq \frac{\int_0^1 (1-s)^{\frac{1}{2}} ds}{\frac{1}{2}\Gamma(\frac{3}{2})} + \frac{\Gamma(\frac{3}{2})}{\frac{1}{2}} \leq 3.1601 \dots$$

which implies that

$$2kG^* \leq (2)(\frac{1}{11})(3.1601) = 0.5745 \dots < 1.$$

And

$$2k\left(\frac{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1) + \Gamma(2-\beta)}{\Gamma(3-\alpha)(\Gamma\alpha-\beta+1)}\right) = \frac{2}{11}\left(\frac{\Gamma(\frac{3}{2})\Gamma(2) + \Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2})\Gamma(2)}\right) = 0.3636 \dots < 1.$$

Or

$$\max\{2kG^*, \frac{2k(\Gamma(3-\alpha)\Gamma(\alpha-\beta+1) + \Gamma(2-\beta))}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)}\} < 1.$$

Hence by Banach contraction mapping the given fractional bounded value problem has a unique solution on $J \in [0, 1]$.

Conflict of interests. The authors declare that they have no competing interests.

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